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LOCALLY FINITE LEAVITT PATH ALGEBRAS

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ABSTRACT

A group-graded K-algebra $A = \bigoplus_{g\in G} A_g$ is called **locally finite** in case each graded component A_g is finite dimensional over K. We characterize the graphs E for which the Leavitt path algebra $L_K(E)$ is locally finite in the standard $\mathbb{Z}$-grading. For a locally finite $\mathbb{Z}$-graded algebra A we show that, if every nonzero graded ideal has finite codimension in A, then every nonzero ideal has finite codimension in A; that is, **$\mathbb{Z}$-graded just infinite** implies **just infinite**. We use this result to characterize the finite graphs E for which the Leavitt path algebra $L_K(E)$ is locally finite just infinite. We then give an explicit description of the graphs and algebras which arise in this way. In particular, we show that the locally finite Leavitt path algebras are precisely the noetherian Leavitt path algebras.

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Throughout this article K will denote a field. As in [8], we say that a K-algebra B is **just infinite dimensional** (or, more concisely, **just infinite**) in case $\dim_K(B)$ is infinite, but $\dim_K(B/I)$ is finite for every nonzero ideal I of B. We say that a $\mathbb{Z}$-graded K-algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is **locally finite** in case $\dim_K(A_n) < \infty$ for every $n \in \mathbb{Z}$.

For a graph E and a field K, the Leavitt path algebra $L_K(E)$, and its analytic counterpart the Cuntz–Krieger graph C*-algebra $C^*(E)$, have been the focus of much recent attention (see, e.g., [1], [4], [6], and [11]). After presenting a brief overview of this topic, in Section 1 we classify those graphs E for which the Leavitt path algebra $L_K(E)$ is locally finite (Theorem 1.8). With a general result about $\mathbb{Z}$-graded algebras in hand, in Section 2 we then find those finite graphs for which $L_K(E)$ is locally finite just infinite (Theorem 2.7). In the final section we describe explicitly those K-algebras which arise in this way. Specifically, in Theorem 3.3, we describe all locally finite just infinite Leavitt path algebras, and then in Theorem 3.8, we describe all locally finite Leavitt path algebras.

Of course, any finite dimensional algebra is necessarily locally finite. Thus the work presented in the current article can be viewed as a logical followup to [3], in which the authors completely classify the finite dimensional Leavitt path algebras.

1. Locally finite Leavitt path algebras

The Leavitt path algebra of a graph is defined in [1]. We briefly recall the essentials here. A **(directed) graph** $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called **vertices** and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called **row-finite**. Throughout this paper we will be concerned only with row-finite graphs. If E^0 is finite then, by the row-finite hypothesis, E^1 must necessarily be finite as well; in this case we say simply that E is **finite**. A vertex which emits no edges is called a **sink**. A **path** μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the **source** of μ, $r(\mu) := r(e_n)$ is the **range** of μ, and n is the **length** of μ. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n\geq 0} E^n$ the set of all paths.

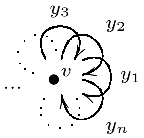
We define the **Leavitt path K-algebra** $L_K(E)$ as the K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 | s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

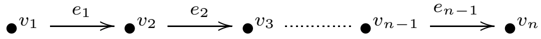
The elements of E^1 are called **real edges**, while for $e \in E^1$ we call e^* a **ghost edge**. The set $\{e^* : e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

It is sometimes helpful to look at $L_K(E)$ in the following way. If we start with a graph E, we can form a new graph $\hat{E}$ having the same vertex set as E, but in which, for each edge e of E, we add an edge e^* whose orientation is opposite to that of e. Next we form $K\hat{E}$, the (now-standard) **path algebra** of $\hat{E}$ with coefficients in K. Finally, $L_K(E)$ can be viewed as the quotient of $K\hat{E}$ modulo the two sided ideal generated by the relations described in (3) and (4) above.

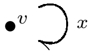
While $L_K(E)$ arises as the quotient of a path algebra, in general these Leavitt path algebras have properties which are not found in the usual path algebra construction. For instance, the class of Leavitt path algebras includes the algebras described by Leavitt in [10]. These algebras (denoted $A = L_K(1, n)$, for each $n \geq 2$) have the property that $A \cong A^n$ as free left A-modules; in particular, they do not possess the Invariant Basis Number property, and so intuitively should be thought of as being "far" from possessing any sort of chain condition. In the more general context of Leavitt path algebras, the Leavitt algebra $L_K(1, n)$ arises as the algebra $L_K(E)$, where E is the "rose with n petals" graph



On the other side of the structural spectrum, the full $n \times n$ matrix ring over K arises as the Leavitt path algebra of the oriented n-line graph



while the Laurent polynomial ring $K[x, x^{-1}]$ arises as the Leavitt path algebra of the "one vertex, one loop" graph



In [3] the authors determine completely the structure of the finite dimensional Leavitt path algebras. It is the goal of the current article to extend these results to the locally finite Leavitt path algebras.

Note that if E is a finite graph then we have $\sum_{v\in E^0} v = 1$; otherwise, by [1, Lemma 1.6], $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices. Conversely, if $L_K(E)$ is unital, then E^0 is finite. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L_K(E)$ generated by H.

It is shown in [1] that $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* : p, q \text{ are paths in } E\}$. In particular, for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is spanned by elements of the form $\{pq^* : \text{length}(p) - \text{length}(q) = n\}$. The degree of an element x, denoted $deg(x)$, is the lowest number n for which $x \in \bigoplus_{m\leq n} L_K(E)_m$. The set of **homogeneous elements** is $\bigcup_{n\in\mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be n-**homogeneous** or **homogeneous of degree** n.

The K-linear extension of the assignment $pq^* \mapsto qp^*$ (for p, q paths in E) yields an involution on $L_K(E)$, which we denote simply as *. Clearly, $(L_K(E)_n)^* = L_K(E)_{-n}$ for all $n \in \mathbb{Z}$.

We will analyze the structure of various graphs in the sequel. An important role is played by the following three concepts. An edge e is an **exit** for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a **closed path based at** v. We denote by $CP_E(v)$ the set of closed paths in E based at v. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a **cycle**.

Definition 1.1: We say that a graph E satisfies Condition (NE) if no cycle in E has an exit.

Of course the one vertex, one loop graph satisfies Condition (NE). More generally, a graph containing one loop together with any number of paths having range equal to the vertex of the loop, but in which no path has source equal to the vertex of the loop, satisfies Condition (NE).

Lemma 1.2: *If a finite graph E satisfies Condition* (NE) *then every path in E of length at least* $\mathrm{card}(E^0)$ *ends in a cycle.*

Proof. Clearly a path μ of length greater than $\mathrm{card}(E^0)$ contains a closed path $\nu = e_1 \dots e_r$. Since E satisfies Condition (NE), then ν must be in fact a cycle; moreover μ necessarily ends in $e_s \dots e_r \dots e_{s-1}$ for some $s \in \{1, \dots, r\}$ because this cycle has no exits. ∎

Let $n \in \mathbb{Z}$. For $m \in \mathbb{N}$ with $n \leq m$, we let C_m^n denote the following subset of the graded component $L_K(E)_n$ of $L_K(E)$:

$$C_m^n = \{pq^* :\ p \in E^m, q \in E^{m-n}\}.$$

For $n > m$, define $C_m^n = \emptyset$.

Lemma 1.3: *Let $n \in \mathbb{Z}$. If there exists $t \in \mathbb{N}$, $t \geq n$, such that $C_{t+1}^n \subseteq \bigcup_{i=1}^t C_i^n$, then $\bigcup_{i=1}^\infty C_i^n = \bigcup_{i=1}^t C_i^n$.*

Proof. Suppose $C_{t+1}^n \subseteq \bigcup_{i=1}^t C_i^n$. By induction on r, we show that for every $r \in \mathbb{N}$, $C_{t+r}^n \subseteq \bigcup_{i=1}^t C_i^n$. The case $r = 1$ is our hypothesis. Suppose $C_{t+r-1}^n \subseteq \bigcup_{i=1}^t C_i^n$, and consider $\mu = e_{t+r}e_{t+r-1} \dots e_1 f_1^* \dots f_{t-n+r-1}^* f_{t-n+r}^* \in C_{t+r}^n$. If we define $\nu = e_{t+r-1} \dots e_1 f_1^* \dots f_{t-n+r-1}^* \in C_{t+r-1}^n$, then $\mu = e_{t+r}\nu f_{t-n+r}^* \in e_{t+r}C_{t+r-1}^n f_{t-n+r}^* \subseteq e_{t+r}(\bigcup_{i=1}^t C_i^n) f_{t-n+r}^* \subseteq \bigcup_{i=2}^{t+1} C_i^n \subseteq \bigcup_{i=1}^t C_i^n$. ∎

We are now in position to obtain the main result of this section.

Theorem 1.4: *For a finite graph E the following conditions are equivalent:*

(i) *$L_K(E)_n$ has infinite dimension for some $n \in \mathbb{Z}$.*
(ii) *$L_K(E)_n$ has infinite dimension for every $n \in \mathbb{Z}$.*
(iii) *There exists a cycle in E with an exit.*

Proof. (ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (iii). Let $n \in \mathbb{Z}$ be such that $L_K(E)_n$ has infinite dimension. Suppose that no cycle has an exit. Let $t = \max(n, \mathrm{card}(E^0))$. We show that $C_{2t+1}^n \subseteq \bigcup_{i=1}^{2t} C_i^n$.

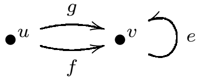
Let ν be a nonzero element in C_{2t+1}^n, say $\nu = e_1 \dots e_{2t+1} f_1^* \dots f_{2t-n+1}^*$. Then, by Lemma 1.2, $r(e_{2t+1})$ is in a cycle c. By noting that $2t-n+1 = t+t-n+1 \geq t+1 \geq \mathrm{card}(E^0)$, Lemma 1.2 can be applied to $f_{2t-n+1} \dots f_1$, so that f_1 must belong to a cycle d. Moreover, since $\nu \neq 0$ and E satisfies Condition (NE), $c = d$ and therefore $e_{2t+1} = f_1$ (by Condition (NE) again). This yields that

$\nu \in C_{2t}^n$, since Condition (NE) implies that $e_{2t+1}f_1^* = s(e_{2t+1})$ (by relation (4)). Now, since $\bigcup_{i=1}^{\infty} C_i^n$ is a generating set for $L_K(E)_n$, by Lemma 1.3 we obtain a contradiction and finish the proof.

(iii) $\Longrightarrow$ (ii). Let f be an exit for a cycle c and suppose that $v := s(f) = s(c)$. Let $k = deg(c)$, and write $c = e_k \dots e_1$. Consider $n \geq 0$ and decompose $n = bk + s$, with $0 \leq s < k$. We claim that $\{e_s \dots e_1 c^b c^r (c^*)^r : r \in \mathbb{N}\}$ is a linearly independent set in $L_K(E)_n$. Indeed, suppose $\sum_{r=i}^n k_r e_s \dots e_1 c^b c^r (c^*)^r = 0$ for $k_r \in K$ with $k_i \neq 0$. Multiply on the left by $(c^*)^i (c^*)^b e_1^* \dots e_s^*$, and on the right by c^i, to get $k_i v + \sum_{r=i+1}^n k_r c^{r-i}(c^*)^{r-i} = 0$ (apply [1, Lemma 2.2]). Since f is an exit for c, we obtain $0 = k_i f^* v + \sum_{r=i+1}^n k_r f^* c^{r-i}(c^*)^{r-i} = k_i f^*$, a contradiction. The case $n < 0$ can be obtained by using the involution: since $L_K(E)_n = (L_K(E)_{-n})^*$, then $\dim_K(L_K(E)_n) = \dim_K((L_K(E)_{-n})^*) = \dim_K(L_K(E)_{-n}) = \infty$. ∎

Remark 1.5: The finiteness hypothesis on E in the preceding result cannot be dropped. For instance, if E is an acyclic graph with infinitely many vertices and only a finite number of edges, then $\dim_K(L_K(E)_0) = \infty$, while $\dim_K(L_K(E)_n) = 0$ for any sufficiently large n. In general this happens for any infinite graph such that $E^n = \emptyset$ for some $n \in \mathbb{N}$.

Remark 1.6: As a consequence of the previous result, for a finite graph E, if one homogenous component of the Leavitt path algebra $L_K(E)$ has infinite dimension then all the homogenous components have that same (necessarily countably infinite) dimension. However, in case the homogeneous components have finite dimension, the dimension of the components can differ. Of course any nontrivial finite dimensional Leavitt path algebra will have this property (see, e.g., [3]). For an infinite dimensional but locally finite example of this phenomenon, consider the Leavitt path algebra of the graph E



Using Lemma 1.3, a straightforward computation yields $\dim_K(L_K(E)_0) = 8$, $\dim_K(L_K(E)_1) = \dim_K(L_K(E)_{-1}) = 9$, and $\dim_K(L_K(E)_n) = 3$ for all n having $|n| \geq 2$.

The following Lemma will be useful throughout the sequel, and will be used often without explicit mention.

LEMMA 1.7: *If $L_K(E)$ is a locally finite Leavitt path algebra, then E is finite.*

Proof. If E were not finite, then E^0 would yield an infinite set of linearly independent elements of $L_K(E)_0$. ∎

With this lemma, Theorem 1.4 gives the following.

THEOREM 1.8: *For a graph E and any field K the following are equivalent:*

(i) $L_K(E)$ *is locally finite.*
(ii) E *is finite and has Condition* (NE)*.*

We end this section by identifying the locally finite simple Leavitt path algebras.

COROLLARY 1.9: *Let $L_K(E)$ be a locally finite Leavitt path algebra. Then $L_K(E)$ is simple if and only if $L_K(E) \cong \mathbb{M}_n(K)$ for some positive integer n.*

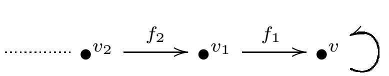
Proof. By Theorem 1.8 we have that E is finite and has Condition (NE). On the other hand, by [1, Theorem 3.11], E has also Condition (L), therefore E is an acyclic graph. Now using [3, Proposition 3.5], $L_K(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$. But $L_K(E)$ being simple implies the desired result. ∎

2. Locally finite just infinite Leavitt path algebras.

Having identified the locally finite Leavitt path algebras in the previous section, we now determine which of these algebras are in fact just infinite.

As we show in the following example, a graded just infinite Leavitt path algebra need not be just infinite. To demonstrate this, we need some additional information about various subsets of vertices of a graph. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called **hereditary** if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is **saturated** if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0.

Example 2.1: The Leavitt path algebra $L_K(E)$ of the following graph E



is a graded simple (and therefore graded just infinite) K-algebra, but is not just infinite, as follows.

It is straightforward to show that the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0. Thus [4, Theorem 6.2] applies, and we conclude that $L_K(E)$ is graded simple (i.e., the only graded ideals of $L_K(E)$ are $\{0\}$ and $L_K(E)$). In particular, $L_K(E)$ is graded just infinite.

We now show that there are ideals in $L_K(E)$ of infinite codimension, specifically, the ideal $I := \langle v+e \rangle$ is such. We start by showing that $v \notin I$. Suppose on the contrary that $v \in I$. Then there exist monomials a_i and b_i in $L_K(E)$, and scalars $k_i \in K$ such that $v = \sum_i k_i a_i (v+e) b_i$. With the use of the last equation and the fact that $v(v+e)v = v+e$ we can assume that $va_iv = a_i$ and $vb_iv = b_i$. Let a denote any element of the form $a = a_i$ or b_i. Write $a = \alpha\beta^*$, where α, β are paths of arbitrary length (they can be of length zero, that is, a vertex). As α is a monomial we necessarily have that v is a local unit for α on the right, so that $a = \alpha v \beta^*$. In this situation we have $v\alpha v = \alpha$ and $v\beta^* v = \beta^*$, so that, $\alpha, \beta \in CP(v)$. But it is clear that in E we have $CP(v) = \{e^k : \ k \geq 0\}$. Thus $a = e^n (e^*)^m$ for some $n, m \geq 0$. Moreover, since e has no exits, then $a = e^n$ or $a = (e^*)^n$, where $n \in \mathbb{N}$.

A similar equation holds for all the monomials a_i and b_i, so that we get $v = \sum_i k_i e^{m_i} (v+e) e^{n_i}$ with m_i and $n_i \in \mathbb{Z}$ (where for $r < 0$ we interpret e^r as $(e^*)^{-r}$). Then $v = \sum_i k_i (e^{m_i+n_i} + e^{m_i+n_i+1})$. Rewrite v as $v = \sum_{i=r}^{s} k_i' (e^i + e^{i+1})$. A degree argument on the highest and on the lowest power of the right hand side shows $s+1 = 0$ and $r = 0$, a contradiction since $r \leq s$. We conclude that $v \notin I$.

Now, consider the infinite set of vertices $\{v_i\}$. We claim that $\{\overline{v_i}\}_i$ is a linearly independent set in $L_K(E)/I$. Suppose otherwise, that for some scalars $k_1, \ldots, k_m \in K$, and some vertices $v_{i_1}, \ldots, v_{i_m}$, $x = \sum_{j=1}^{m} k_j v_{i_j} \in I$. Consider $w = v_{i_j} \in \{v_{i_1}, \ldots, v_{i_m}\}$, with $k_j \neq 0$. Then $v = f_1^* \ldots f_{i_j}^* f_{i_j} \ldots f_1 = f_1^* \ldots f_{i_j}^* w f_{i_j} \ldots f_1 = {k_j}^{-1} f_1^* \ldots f_{i_j}^* w x w f_{i_j} \ldots f_1 \in I$, contrary to the result of the previous paragraph.

Here is some additional useful information about graphs. If μ is a path having $v = s(\mu) = r(\mu)$ and $s(\mu_i) \neq v$ for every $i > 1$, then μ is called a **closed simple path based at** v. We denote by $CSP_E(v)$ the set of closed simple paths in E based at v. For a path μ we denote by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) : i = 1, \ldots, n\}$. For a graph E, we let V_0 denote the set of vertices which do not lie on any cycle (see [2]), i.e.

$$V_0 = \{v \in E^0 : CSP(v) = \emptyset\}.$$

For an $H \in \mathcal{H}$, the **quotient graph** of E by H is given by

$$E/H = (E^0 - H, \{e \in E^1 : r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}).$$

Lemma 2.2: *If $L_K(E)$ is a graded just infinite Leavitt path algebra and $\emptyset \neq H \in \mathcal{H}$ then $E^0 - H$ is a finite set and $E^0 - V_0 \subseteq H$.*

Proof. If $E^0 - H$ were infinite then E/H would contain infinitely many vertices and $L_K(E/H)$ would be infinite dimensional but, by [5, Lemma 2.3 (1)], $L_K(E/H) \cong L_K(E)/I(H)$ with $I(H)$ a nonzero graded ideal of $L_K(E)$, which is impossible by the hypothesis.

Suppose now that there exists $v \in (E^0 - V_0) - H$, that is, there exists a cycle μ based at $v \notin H$. As H is hereditary, $\mu^0 \cap H = \emptyset$. If we write $\mu = \mu_1 \dots \mu_n$, then $\mu_i \in E/H$ as $r(\mu_i) \notin H$. Thus E/H completely contains the cycle μ, and therefore again $L_K(E/H)$ is infinite dimensional, contrary to the hypothesis. ∎

Lemma 2.3: *Let $L_K(E)$ be a graded just infinite Leavitt path algebra. If $H, H' \in \mathcal{H}$ are nonempty, then the intersection $H \cap H'$ is nonempty.*

Proof. Since $L_K(E)$ is infinite dimensional, by [3, Corollary 3.6], either E^0 is infinite or E is not acyclic. In the first case, apply Lemma 2.2 to obtain that both $E^0 - H$ and $E^0 - H'$ are finite. Now if $H \cap H' = \emptyset$, then $H \subseteq E^0 - H'$, and therefore both H and $E^0 - H$ are finite sets, which cannot happen when E^0 is infinite. Now, if E is not acyclic, then pick any cycle in E, and let v denote the vertex at which the cycle is based. But then $v \notin V_0$, and again Lemma 2.2 applies to get $v \in H \cap H'$. ∎

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is **cofinal** if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$. We say that a graph E is **cofinal** if so are all the vertices of E.

The set $T(v) = \{w \in E^0 : v \geq w\}$ is the **tree** of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. The **hereditary saturated closure** of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [4] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

(1) $\Lambda_0(X) = T(X)$,
(2) $\Lambda_n(X) = \{y \in E^0 : s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

We now have the tools to give a graph-theoretic characterization of all of the graded just infinite Leavitt path algebras.

THEOREM 2.4: *Let $L_K(E)$ be an infinite dimensional Leavitt path algebra. The following conditions are equivalent:*

(i) *$L_K(E)$ is graded just infinite.*
(ii) *E is cofinal.*
(iii) *$L_K(E)$ is graded simple.*

Proof. (ii) $\Longrightarrow$ (iii). By [4, Theorem 6.2] the result follows.

(iii) $\Longrightarrow$ (i) is evident since by hypothesis $L_K(E)$ is infinite dimensional.

(i) $\Longrightarrow$ (ii). If we suppose that E is not cofinal, then by again using [5, Lemma 2.8] there exists a nontrivial hereditary and saturated subset H of E^0. Let y_1 denote a vertex which is not in H, and consider $H' = \overline{\{y_1\}}$. By Lemma 2.3, $H \cap H' \neq \emptyset$. In this case the hereditary saturated closure described above gives us some minimal $n \in \mathbb{N}$ with $H \cap \Lambda_n(\{y_1\}) \neq \emptyset$.

If $n > 0$, then we have that $H \cap \{y \in E^0 : \emptyset \neq r(s^{-1}(y)) \subseteq \Lambda_{n-1}(\{y_1\})\} \neq \emptyset$, since $H \cap \Lambda_{n-1}(\{y_1\}) = \emptyset$.

Take $z \in H$ with $\emptyset \neq r(s^{-1}(z)) \subseteq \Lambda_{n-1}(\{y_1\})$. In particular $r(s^{-1}(z)) \cap H = \emptyset$, which contradicts that H is hereditary.

So $n = 0$, and therefore $H \cap T(\{y_1\}) \neq \emptyset$. Since $y_1 \notin H$, we can then find a path $\nu = \nu_1 \dots \nu_n$ with $n \geq 1$ such that $s(\nu) = y_1, r(\nu) \in H$ but $r(\nu_i) \notin H$ for $i < n$. Since H is saturated and $s(\nu_n) = r(\nu_{n-1}) \notin H$, there must exist $e \in E^1$ with $r(e) \notin H$ and $s(e) = s(\nu_n)$. We claim that $r(e) \neq s(\nu_i)$ for every $i = 1, \dots, n$; otherwise, if $r(e) = s(\nu_i)$ for some i, then $s(\nu_i) \notin V_0$ as the path given by $\nu_i \nu_{i+1} \dots \nu_{n-1} e$ is a closed path based at this vertex, which yields a cycle based at this vertex, but this contradicts, by Lemma 2.2, the fact that $s(\nu_i) \notin H$.

Rename this new vertex $r(e)$ as y_2. In particular $y_1 \neq y_2$. Repeat the process with y_2, thus yielding a path $\delta = \delta_1 \dots \delta_m$ with $m \geq 1$ such that

$s(\delta) = y_2, r(\delta) \in H$ and $r(\delta_i) \notin H$ for $i < m$. Once more, there exists, by the saturation of H, an edge $f \in E^1$ with $r(f) \notin H$ and $s(f) = s(\delta_m)$. Not only do we have $r(f) \neq s(\delta_i)$ for all $i = 1, \ldots, m$ as before, but also $r(f) \neq s(\nu_i)$ for $i = 1, \ldots, n$. (Otherwise, if for instance $r(f) = s(\nu_1) = y_1$, then $\nu_1 \ldots \nu_{n-1} e \delta_1 \ldots \delta_{m-1} f$ is a closed path based at $r(f) \notin H$, a contradiction to Lemma 2.2.)

Continuing in this way, we rename $r(f)$ as y_3, so that in particular we have $y_3 \neq y_1, y_2$. In this way we obtain an infinite sequence $\{y_i\}_{i=1}^{\infty} \subseteq E^0 - H$, which cannot happen by Lemma 2.2. This finishes the proof. ∎

We complete this section by showing that for locally finite Leavitt path algebras, the property of being graded just infinite implies, in fact, that the algebra is just infinite. To do so, we prove a general result about all $\mathbb{Z}$-graded algebras. By [12, Lemma 1.3(d)], if A is a locally finite **positively** $\mathbb{Z}$-graded algebra (i.e. $A_n = \{0\}$ for all $n < 0$), then A is just infinite in case it is graded just infinite. We extend this result to all $\mathbb{Z}$-graded algebras. Our approach is largely based on an idea presented by D. Rogalski in a private communication.

Proposition 2.5: *Let A be a locally finite $\mathbb{Z}$-graded K-algebra. Then A is graded just infinite if and only if A is just infinite.*

Proof. Suppose that the algebra A is graded just infinite, and let L be a nonzero ideal of A. We note that the quotient algebra A/L is generated by homogeneous elements as a K-vector space. Pick any nonzero element $x \in L$, and write $x = \sum_{i=m}^{n} x_i$, with $x_i \in A_i$, and $x_m, x_n \neq 0$. In particular, $x \in \bigoplus_{i=m}^{n} A_i$.

Since Ax_nA is a nonzero graded ideal of A, by hypothesis Ax_nA has finite codimension in A, so that there exists $r \in \mathbb{N}$ such that $A_i \subseteq Ax_nA$ for every $i \in \mathbb{Z}$ having $|i| > r$. Analogously, there exists $s \in \mathbb{N}$ such that $A_i \subseteq Ax_mA$ for every $i \in \mathbb{Z}$ having $|i| > s$. Define $p = \max\{r, s, n-m\}$. We show that A/L is in fact generated by elements of the form $\{\overline{y_i} : y_i \in A_i, -p \leq i \leq p\}$. As A is locally finite, this will yield the desired result.

For any $j > p$ consider $y_j \in A_j$. Then $y_j \in Ax_nA$, so we can write $y_j = \sum_t a_{\sigma_t} x_n b_{\tau_t}$ with $a_{\sigma_t} \in A_{\sigma_t}$ and $b_{\tau_t} \in A_{\tau_t}$. Note that $\sigma_t + n + \tau_t = j$. For each i with $m \leq i \leq n$ define $c_{j-n+i} = \sum_t a_{\sigma_t} x_i b_{\tau_t}$, and then define $z = \sum_{i=m}^{n} c_{j-n+i}$. Then $z = \sum_t a_{\sigma_t} x b_{\tau_t}$, so $z \in L$. Therefore in A/L we have $\overline{y_j} = -(\overline{z - y_j})$. But $z - y_j$ has homogenous components of degree $j-(n-m)$ through $j-1$. Therefore, since $j > n-m$, all these degrees are positive. Thus, modulo L, we have written y_j as the sum $\sum_{i=q_1}^{j_1} c_i$, where $c_i \in A_i$, each i is positive, and $j_1 < j$.

If $j_1 < p$ we stop. If not, repeat the above process on c_{j_1}. Specifically, we are able to express c_{j_1} as a sum of homogeneous components of degree less than j_1, but all of them positive.

In this way, after at most $j-p$ steps, we will have written y_j (modulo L) as a sum of homogeneous elements of positive degree less than p. That is, $(A_j + L)/L \subseteq \bigoplus_{i=1}^{p}(A_i + L)/L$ for all $j > p$.

A completely analogous argument yields that for any $j < -p$ we have $(A_j+L)/L \subseteq \bigoplus_{i=-p}^{-1}(A_i+L)/L$. This then yields that $A/L \subseteq \bigoplus_{i=-p}^{p}(A_i+L)/L$. Since each A_i is finite dimensional, we are done. ∎

As an easy consequence of Proposition 2.5, we get the following well-known result (see, e.g., [7]). The algebras which appear in this result will play a central role in the sequel.

Corollary 2.6: *Let K be a field, and let A denote the Laurent polynomial ring $A = K[x, x^{-1}]$. Then for any $n > 0$ the matrix ring $\mathbb{M}_n(A)$ is just infinite.*

Proof. As $\dim_K(A_i) = 1$ for all $i \in \mathbb{Z}$, A is locally finite. Since every nonzero homogeneous element of A is invertible, A is graded simple, so is, in particular, graded just infinite. Now apply Proposition 2.5 to conclude that A is just infinite. Since matrix rings over just infinite rings are again just infinite (see e.g. [7, Lemma 1(i)]) we are done. ∎

Theorem 2.4 together with Proposition 2.5 now immediately yield the result about locally finite just infinite Leavitt path algebras which was mentioned in the introduction.

Theorem 2.7: *Let E be a graph such that $L_K(E)$ is infinite dimensional and locally finite. Then the following conditions are equivalent:*

(i) *$L_K(E)$ is graded just infinite.*
(ii) *E is cofinal.*
(iii) *$L_K(E)$ is graded simple.*
(iv) *$L_K(E)$ is just infinite.*

As noted previously in Example 2.1, the local finiteness condition on E (which implies the finiteness of E) cannot be dropped in Theorem 2.7.

3. Explicit graph-theoretic and algebraic descriptions of locally finite Leavitt path algebras, and of locally finite just infinite Leavitt path algebras

In this final section of the article we achieve three goals. First, we describe concretely the graphs which arise in Theorem 2.7. Consequently, we obtain the isomorphism classes of the locally finite just infinite Leavitt path algebras (Theorem 3.3). Building on these ideas, we are able to describe the isomorphism classes of all locally finite Leavitt path algebras (Theorem 3.8). As a result, we conclude in Theorem 3.10 that the locally finite Leavitt path algebras are precisely the noetherian Leavitt path algebras.

Definition 3.1: We say that a graph E is a C_n**-comet** if it is finite, has exactly one cycle C_n (this unique cycle contains n vertices), and $T(v) \cap (C_n)^0 \neq \emptyset$ for every vertex $v \in E^0$.

PROPOSITION 3.2: *The Leavitt path algebra $L_K(E)$ is locally finite just infinite if and only if E is a C_n-comet.*

Proof. Suppose that E is a C_n-comet. By hypothesis, E is finite and contains a cycle. Then $L_K(E)$ is infinite dimensional. Locally finiteness follows from the fact that the cycle C_n has no exits and an application of Theorem 1.8.

Now let $v \in E^0$, and consider the hereditary and saturated closure $\overline{\{v\}}$. By hypothesis we have $(C_n)^0 \cap \overline{\{v\}} \neq \emptyset$, and also by hereditariness $(C_n)^0 \subseteq \overline{\{v\}}$. Just suppose that $\overline{\{v\}} \neq E^0$. Then take $y_1 \notin \overline{\{v\}}$. As E is a C_n-comet we get $\overline{\{v\}} \cap T(\{y_1\}) \neq \emptyset$. Since $y_1 \notin \overline{\{v\}}$, we can then find a path $\nu = \nu_1 \dots \nu_n$ with $n \geq 1$ such that $s(\nu) = y_1, r(\nu) \in \overline{\{v\}}$ but $r(\nu_i) \notin \overline{\{v\}}$ for $i < n$. If we focus on $s(\nu_n)$, since $\overline{\{v\}}$ is saturated and $s(\nu_n) \notin \overline{\{v\}}$, there must exist $e \in E^1$ with $r(e) \notin \overline{\{v\}}$ and $s(e) = s(\nu_n)$. We claim that $r(e) \neq s(\nu_i)$ for every $i = 1, \dots, n$. Otherwise, if $r(e) = s(\nu_i)$ for some i, then $s(\nu_i) \notin V_0$ as the path given by $\nu_i \nu_{i+1} \dots \nu_{n-1} e$ is a closed path based at this vertex, but then that would imply the existence of a cycle contained in $E^0 - \overline{\{v\}}$, contradicting the fact that C_n is the only cycle in E.

Rename this newly obtained vertex $r(e)$ by y_2. In particular $y_1 \neq y_2$. Repeat the process with y_2 so that we can find a path $\delta = \delta_1 \dots \delta_m$ with $m \geq 1$ such that $s(\delta) = y_2, r(\delta) \in \overline{\{v\}}$ and $r(\delta_i) \notin \overline{\{v\}}$ for $i < m$. Once more, there exists, by the saturation of $\overline{\{v\}}$, an edge $f \in E^1$ with $r(f) \notin \overline{\{v\}}$ and $s(f) = s(\delta_m)$. Not only do we have $r(f) \neq s(\delta_i)$ for all $i = 1, \dots, m$ as before, but also

$r(f) \neq s(\nu_i)$ for $i = 1, \dots, n$. (If, for instance, we have $r(f) = s(\nu_1) = y_1$, then $\nu_1 \dots \nu_{n-1} e \delta_1 \dots \delta_{m-1} f$ is a closed path based at $r(f) \notin \overline{\{v\}}$, a contradiction again).

Write then $y_3 = r(f)$, so that in particular we have $y_3 \neq y_1, y_2$. In this way we obtain an infinite sequence $\{y_i\}_{i=1}^{\infty} \subseteq E^0 - \overline{\{v\}}$, which cannot happen as E is finite. Now [5, Lemma 2.8] applies to yield the cofinality of E. Now, Theorem 2.7 finishes the proof.

Conversely, since $L_K(E)$ is locally finite, we have in particular that E is finite. But since $L_K(E)$ is just infinite we have in particular that $L_K(E)$ is infinite dimensional, so that by [3, Corollary 3.6] we have that E contains a cycle C_n. Consider the case $v \notin (C_n)^0$. Use Theorem 2.7 to receive that E is cofinal, and by [5, Lemma 2.8], $\overline{\{v\}} = E^0$. Let t denote the smallest non-negative integer having $\Lambda_t(\{v\}) \cap (C_n)^0 \neq \emptyset$. Pick w in this intersection. If $t > 0$, then $\Lambda_{t-1}(\{v\}) \cap (C_n)^0 = \emptyset$, and therefore $\emptyset \neq r(s^{-1}(w)) \subseteq \Lambda_{t-1}(\{v\})$. In particular, $\Lambda_{t-1}(\{v\}) \cap (C_n)^0 \neq \emptyset$, a contradiction, so t must be zero, thus $T(\{v\}) \cap (C_n)^0 \neq \emptyset$. This also shows that C_n is the only cycle, because the existence of any other cycle in E would necessarily yield an exit for C_n, which cannot happen by Theorem 1.8. ∎

THEOREM 3.3: *Let E be a graph for which $L_K(E)$ is locally finite just infinite. Let C denote the unique cycle in E, and let v be any vertex in C. Then*

$$L_K(E) \cong \mathbb{M}_n(K[x, x^{-1}]),$$

where n is the number of paths in E which do not contain C, and which end in v. In particular, $L_K(E) \cong L_K(C_n)$.

Proof. By Proposition 3.2 there is an integer m so that graph E is a C_m-comet, so that $C = C_m$ is the unique cycle in E. Let $e_1, \dots, e_m$ and $v_1, \dots, v_m$ be respectively the edges and the vertices of the cycle C_m. That is: $r(e_i) = v_i$ for all i, $s(e_i) = v_{i-1}$ for $i > 1$, and $s(e_1) = v_m$. We eliminate the edge e_1 in the graph E, and denote the resulting graph by F.

Let $P = \{p_i : 1 \leq i \leq n\}$ denote the set of all paths which end in v_m, and which do not contain the cycle C_m. That is, p_i's are the paths in F ending in v_m, or with the notation used in [3], p_i's are the paths in $R(v_m)$. Since E is a C_m-comet graph, the graph F is finite and acyclic, so that $|P| = n$ is indeed finite.

Consider the set $\mathcal{B} = \{p_i c^k p_j^*\}_{i,j\in\{1,\ldots,n\},k\in\mathbb{Z}}$, where $c = e_1 \ldots e_m$ is the cycle C_m. (We use the notation $c^k = (c^*)^{-k}$ for negative k, and that $c^0 = v_m$. Note that these conventions are possible as the usual rules for exponents are valid here, due to the fact that the cycle C_m has no exits.)

We claim that $\mathcal{B}$ is a basis of $L_K(E)$ as a K-vector space. To this end, we first consider the inclusion map from F to E. This map is a complete graph homomorphism (see [4, p. 5]), and therefore induces a K-algebra homomorphism $\varphi : L_K(F) \to L_K(E)$ by [4, Lemma 2.2] since the relations (1) through (4) in $L_K(F)$ are preserved by φ. Moreover, F has v_m as its only sink, as every other vertex connects to the cycle C_m and therefore to v_m.

Thus, by [3, Proposition 3.5], $L_K(F)$ is simple and therefore φ is a monomorphism. If fact, it was shown in [3, Proof of Lemma 3.4] that $\{p_i p_j^*\}_{i,j\in\{1,\ldots,n\}}$ is a set of matrix units such that $p_i^* p_j = \delta_{ij} v_m$. We translate this information via the monomorphism φ to get the analogous relations in $L_K(E)$.

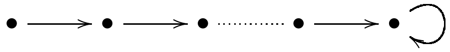
Suppose now that $x = \sum_{i,j,k} \alpha_{ijk} p_i c^k p_j^* = 0$ for $\alpha_{ijk} \in K$. Then for arbitrary i_0, j_0 we have that $0 = p_{i_0}^* x p_{j_0} = \sum_{i_0,j_0,k} \alpha_{i_0 j_0 k} c^k$, which then gives $\alpha_{i_0 j_0 k} = 0$ for all $k \in \mathbb{Z}$, as powers of the cycle are linearly independent in $L_K(E)$. This shows that $\mathcal{B}$ is a linearly independent set.

On the other hand, we realize that the set $Y = \{p_i p_j^*\} \cup \{e_1, e_1^*\}$ generates $L_K(E)$ as a K-algebra (to show this it is enough to consider that $L_K(F)$ is generated as a K-algebra by $\{p_i p_j^*\}$ and apply the monomorphism φ). Clearly, $Y \subseteq \mathcal{B}$ (for instance, $e_1 = c(e_2 \ldots e_m)^* \in \mathcal{B}$). Moreover, $\mathcal{B}$ is closed under products with the general formula $(p_i c^k p_j^*)(p_r c^t p_s^*) = \delta_{jr} p_i c^{k+t} p_s^*$. Thus, we have proved that $\mathcal{B}$ is a generator set of $L_K(E)$ as a K-vector space, and therefore, a basis.

Finally, define the map $\phi : L_K(E) \to \mathbb{M}_n(K[x, x^{-1}])$ on the basis by setting $\phi(p_i c^k p_j^*) = x^k e_{ij}$ (where e_{ij} denotes the standard (i, j)-matrix unit), and extend linearly to all of $L_K(E)$. This map is a K-algebra homomorphism as we have $\phi((p_i c^k p_j^*)(p_r c^t p_s^*)) = \phi(\delta_{jr} p_i c^{k+t} p_s^*) = \delta_{jr} x^{k+t} e_{is} = (x^k e_{ij})(x^t e_{rs}) = \phi(p_i c^k p_j^*)\phi(p_r c^t p_s^*)$. It is bijective as it maps a basis of $L_K(E)$ to a basis of $\mathbb{M}_n(K[x, x^{-1}])$. Therefore it is the desired isomorphism. ∎

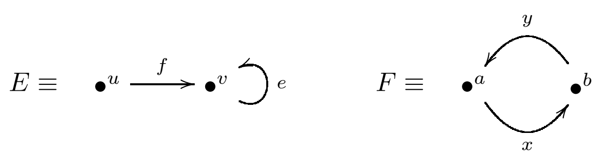
As a specific consequence of Theorem 3.3 we can complete the $n = 1$ case of [2, Proposition 13] .

Corollary 3.4: *Let E_1^n denote the graph with n vertices and n edges*



Then $L_K(E_1^n) \cong \mathbb{M}_n(K[x, x^{-1}])$.

Remark 3.5: It turns out that two nonisomorphic C_m-comets can give rise to isomorphic Leavitt path algebras, although this isomorphism need not be graded. For example, consider the C_1-comet graph E and C_2-comet graph F given by



Theorem 3.3 yields that each is isomorphic to $\mathbb{M}_2(K[x, x^{-1}])$. However, these two Leavitt path algebras cannot be isomorphic as graded algebras, since one can check that $L_K(E)_0$ is generated as a K-vector space by the linearly independent set $\{u, v, ef^*, fe^*\}$, while $L_K(F)_0$ is generated by the linearly independent set $\{a, b\}$, so that $\dim_K L_K(E)_0 \neq \dim_K L_K(F)_0$.

Corollary 3.6: *For* $n, n' \in \mathbb{N}$ *we have that* $L_K(C_n) \cong L_K(C_{n'})$ *if and only if* $n = n'$.

Proof. Since $K[x, x^{-1}]$ is a commutative ring, we may apply [9, Exercise 14, p. 480] together with Theorem 3.3 to get the result. ∎

The corollary in turn gives the following complete classification of the locally finite just infinite Leavitt path algebras.

Corollary 3.7: *A complete irredundant set of the isomorphism classes of locally finite just infinite Leavitt path algebras is given by*

$$\{\mathbb{M}_n(K[x, x^{-1}]) : n \in \mathbb{N}\}.$$

Having described the locally finite just infinite Leavitt path algebras, we are now in position to describe all locally finite Leavitt path algebras. As a consequence of the following theorem, we will see two things. First, that the class of locally finite Leavitt path algebras consists precisely of finite direct sums of locally finite just infinite Leavitt path algebras with finite dimensional Leavitt path algebras. Second, that the locally finite Leavitt path algebras are precisely the noetherian Leavitt path algebras.

THEOREM 3.8: *Let E be a graph such that $L_K(E)$ is a locally finite algebra. Then $L_K(E)$ is isomorphic to*

$$\left(\bigoplus_{i=1}^{l}\mathbb{M}_{m_i}(K[x,x^{-1}])\right)\oplus\left(\bigoplus_{j=1}^{l'}\mathbb{M}_{n_j}(K)\right),$$

where: l is the number of cycles in E (call them $c_1,\dots,c_l$), m_i is the number of paths ending in a fixed (although arbitrary) vertex v_{m_i} of the cycle c_i which do not contain the cycle itself (for $1\le i\le l$); l' is the number of sinks in E (call them $w_{l+1},\dots,w_{l+l'}$), and for every $j\in\{1,\dots,l'\}$, n_j is the number of paths ending in the sink w_{l+j}.

Proof. Let Λ_i be the set of paths in E ending in a fixed vertex v_{m_i} of the cycle c_i which do not contain c_i. Write $c_i=e_1^i\dots e_{m_i}^i$ and $c_i^0=\{v_1^i,\dots,v_{m_i}^i\}$, where $r(e_k^i)=v_k^i$ for all k, $s(e_1^i)=v_{m_i}^i$ and $s(e_k^i)=v_{k-1}^i$ for all $k\ge 1$.

We pull out the edges e_1^i in the graph E to obtain a new graph, which we denote by F.

For a sink w_j with $j=l+1,\dots,l+l'$, let Λ_j be the set of paths of E ending in the sink w_j. Let $\Lambda=\bigcup\Lambda_{i'}=\{p_{j'}\}$. Consider

$$X=\{p_rc_t^kp_s^*:k\in\mathbb{Z};r,s=1,\dots,\operatorname{card}(\Lambda);t=1,\dots,l+l'\},$$

where for $t>l$ we let c_t denote w_t, w_t^k denote w_t for all $k\in\mathbb{Z}$, and c_t^k denote $(c_t^*)^{-k}$ for $k<0$, $t\in\{1,\dots,l\}$.

Let $\mathcal{B}$ be the set of all nonzero elements in X. Note that an element $p_rc_t^kp_s^*$ is in $\mathcal{B}$ if and only if $p_r,p_s\in\Lambda_t$ for $t\in\{1,\dots,l+l'\}$.

We claim that $\mathcal{B}$ is a basis for $L_K(E)$ as a K-vector space. To show this, define the inclusion map $\varphi:L_K(F)\to L_K(E)$ in the natural way. It is a well-defined homomorphism because the relations (1)–(4) in $L_K(F)$ are consistent with those in $L_K(E)$. To show that φ is a monomorphism, we produce a left inverse.

Define $\psi:L_K(E)\to L_K(F)$, first on generators, by setting

$$\psi(e_1^i)=(e_{m_i}^i)^*\dots(e_2^i)^*\quad\text{and}\quad\psi(x)=x\quad\text{for every }x\ne e_1^i$$

and then extending to all of $L_K(E)$. It is long, but straightforward, to check that ψ is well-defined, and $\psi\varphi=1_{L_K(F)}$.

Following [3, Lemma 3.4 and Proposition 3.5], we have that the elements in the set $\{p_ip_j^*:p_i,p_j\in\Lambda_t,\text{ for an arbitrary }t\}$ are a set of matrix units in I_v, for $v\in\{w_{l+1},\dots,w_{l+l'}\}\cup\{v_{m_i}^i,i=1,\dots,l\}$. Hence their union, call it Γ, generates $L_K(F)$. Applying the monomorphism ψ we obtain that $Y = \Gamma \cup \{e_1^i, (e_1^i)^*, i = 1, \ldots, l\}$ generates $L_K(E)$ as a K-algebra. Clearly $Y \subseteq \mathcal{B}$, and Y is closed under products because the general formula

$$(p_i c_\sigma^k p_j^*)(p_r c_\tau^t p_s^*) = \delta_{\sigma\tau}\delta_{jr} p_i c_\sigma^{k+t} p_s^*$$

holds. It can be shown, as in the proof of Theorem 3.3, that $\mathcal{B}$ is a linearly independent set.

Finally, define ϕ as the K-linear extension of:

$$L_K(E) \longrightarrow \left(\bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(K[x, x^{-1}])\right) \oplus \left(\bigoplus_{j=1}^{l'} \mathbb{M}_{n_j}(K)\right)$$

$$\Lambda_k - \{0\} \ni p_i c_k^t p_j^* \longmapsto \begin{cases} x^t p_i p_j^* & \text{for} \quad k = 1, \ldots, l \\ p_i p_j^* & \text{for} \quad k = l+1, \ldots, l+l' \end{cases}$$

This map is a K-algebra homomorphism, and is in fact an isomorphism because it sends a basis of $L_K(E)$ to a basis of $\left(\bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(K[x, x^{-1}])\right) \oplus \left(\bigoplus_{j=1}^{l'} \mathbb{M}_{n_j}(K)\right)$. ∎

The description of the locally finite Leavitt path algebras given in Theorem 3.8 yields the final two results of this article.

Corollary 3.9: *The class of locally finite Leavitt path algebras consists precisely of finite direct sums of locally finite just infinite Leavitt path algebras and finite dimensional Leavitt path algebras.*

Proof. If $L_K(E)$ is locally finite, then by Theorem 3.8 we have

$$L_K(E) \cong \left(\bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(K[x, x^{-1}])\right) \oplus \left(\bigoplus_{j=1}^{l'} \mathbb{M}_{n_j}(K)\right).$$

The result now follows from Theorem 3.3 and [3, Corollary 3.7]. ∎

Theorem 3.10: *For a graph E and field K the following conditions are equivalent:*

(i) *$L_K(E)$ is locally finite.*
(ii) *$L_K(E)$ is left or right noetherian.*
(ii)′ *$L_K(E)$ is left and right noetherian.*
(iii) *E is finite and has Condition* (NE).

Proof. (i) $\Longrightarrow$ (ii)$'$. It is well-known that $A = K[x, x^{-1}]$ is a left and right noetherian ring, and hence so is any finite matrix ring over A. Now the result follows directly from Theorem 3.8.

(ii) $\Longrightarrow$ (iii). It is clear that E must be finite. Suppose to the contrary that there exists a cycle in E with an exit e. Denote $s(e)$ by v, and let μ denote the cycle based at v. We claim that

$$\{0\} \subset L_K(E)(v - \mu\mu^*) \subset L_K(E)(v - \mu^2(\mu^*)^2) \subset \cdots$$

is a properly increasing sequence of left ideals of $L_K(E)$. The containment

$$L_K(E)(v - \mu^i(\mu^*)^i) \subset L_K(E)(v - \mu^{i+1}(\mu^*)^{i+1})$$

for each $i \geq 0$ follows from the easily checked equation

$$v - \mu^i(\mu^*)^i = (v - \mu^i(\mu^*)^i)(v - \mu^{i+1}(\mu^*)^{i+1}).$$

To show that the containments are proper, we show that $v - \mu^{i+1}(\mu^*)^{i+1} \notin L_K(E)(v - \mu^i(\mu^*)^i)$. On the contrary, if $v - \mu^{i+1}(\mu^*)^{i+1} = \alpha(v - \mu^i(\mu^*)^i)$ for some $\alpha \in L_K(E)$, then multiplying on the right by μ^i would give $\mu^i - \mu^{i+1}\mu^* = \alpha(\mu^i - \mu^i) = 0$, so that $\mu^i = \mu^{i+1}\mu^*$, which gives $\mu^i e = \mu^{i+1}\mu^* e$. But this is impossible, as follows. Since $s(e) = r(\mu) = v$ we have $\mu^i e \neq 0$ in $L_K(E)$. But since e is an exit for μ we have $\mu^* e = 0$, so that $\mu^{i+1}\mu^* e = 0$, a contradiction.

(iii) $\Longrightarrow$ (i) follows from Theorem 1.8. ∎

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